

Physics 1A Discussion (Week 4)

Problem 1 (Special Relativistic Doppler Effect)

A person in a spaceship moves at a velocity $-v\hat{x}$ with respect to a stationary observer. (In natural units, v is measured as a fraction of the speed of light and is unitless.) The person holds a laser pointer that emits a photon in the $+\hat{x}$ direction, toward the stationary observer. This is shown in the diagram below.

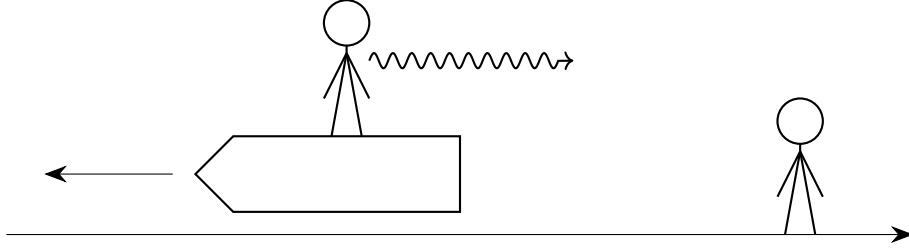


Figure 1: The set up in the observer's rest frame

In the rest frame of the source (frame S), the photon has angular frequency ω . (That means that in natural units, the energy of the photon is ω , since $\hbar$ is set to 1.) In the rest frame of the observer (frame S'), the photon has angular frequency ω' . The observer has mass m .

(a) In frame S , let k^μ be the four-momentum of the photon, and let p^μ be the four-momentum of the observer. Find $k_\mu p^\mu$ in terms of m , ω , and v . Then find $k'_\mu p'^\mu$ in terms of m , ω' , and v . You do not need to use the explicit Lorentz transformation.

The energy of a photon is $E = \hbar\omega$; in natural units, this is $E = \omega$ (more precisely, in natural units, we measure frequency in energy units). The relativistic relationship between energy and momentum is $E^2 = (pc)^2 + (mc^2)^2$; in natural units, this is $E^2 = p^2 + m^2$. For a photon, which is massless, this boils down to $E = |\mathbf{p}|$. Therefore, the four-momentum of a photon with angular frequency ω moving in the $+\hat{x}$ -direction in each frame is

$$k^\mu = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix}, \quad k'^\mu = \begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \omega' \\ \omega' \\ 0 \\ 0 \end{pmatrix} \quad (1)$$

We may convert k^μ and k'^μ to the corresponding dual vectors (row vectors) by adding a minus sign to the spatial components:

$$k_\mu = \eta_{\mu\nu} k^\nu = (\omega \quad -\omega \quad 0 \quad 0), \quad k'_\mu = \eta_{\mu\nu} k'^\nu = (\omega' \quad -\omega' \quad 0 \quad 0) \quad (2)$$

The four-momentum of a massive particle, like the observer, is given in natural units by

$$p^\mu = \begin{pmatrix} \gamma m \\ \gamma m v_x \\ \gamma m v_y \\ \gamma m v_z \end{pmatrix} \quad (3)$$

In frame S , the observer moves at a velocity $+v\hat{x}$ (so $\gamma = (1 - v^2)^{-1/2}$ in frame S), and in frame S' , the observer is at rest (so $\gamma = 1$ in frame S'). This implies that

$$p^\mu = \frac{m}{\sqrt{1 - v^2}} \begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix}, \quad p'^\mu = m \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4)$$

Now, we may take the dot products directly:

$$k_\mu p^\mu = \frac{m}{\sqrt{1-v^2}} (\omega \quad -\omega \quad 0 \quad 0) \begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix} = \frac{m\omega}{\sqrt{1-v^2}} (1-v) \quad (5)$$

$$k'_\mu p'^\mu = m (\omega' \quad -\omega' \quad 0 \quad 0) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = m\omega' \quad (6)$$

(b) We can set $k_\mu p^\mu$ equal to $k'_\mu p'^\mu$. Why?

In special relativity, dot products are always Lorentz invariant, meaning that they are the same in all reference frames. Therefore, we may set $k_\mu p^\mu$ equal to $k'_\mu p'^\mu$. (Indeed, the most fundamental way to define a Lorentz transformation is to say that a Lorentz transformation is a transformation that leaves dot products invariant.)

(c) Set $k_\mu p^\mu$ equal to $k'_\mu p'^\mu$ and solve for ω' in terms of ω . Then use this information to solve for the ratio of the observed to the emitted photon wavelength

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+v}{1-v}} \quad (7)$$

Setting $k_\mu p^\mu$ equal to $k'_\mu p'^\mu$, we get

$$\frac{m\omega}{\sqrt{1-v^2}} (1-v) = m\omega' \quad (8)$$

or

$$\frac{\omega'}{\omega} = \frac{1-v}{\sqrt{1-v^2}} = \frac{1-v}{\sqrt{(1-v)(1+v)}} = \sqrt{\frac{1-v}{1+v}} \quad (9)$$

The relationship between wavelength and angular frequency is $\lambda = 2\pi/k = 2\pi c/\omega$, or $\lambda = 2\pi/\omega$ in natural units. Therefore,

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+v}{1-v}} \equiv 1+z \quad (10)$$

Note that if $v > 0$ (source and observer moving away from each other), $\lambda' > \lambda$, meaning the observed wavelength is *longer* than the emitted wavelength. In the final equation, z is called *redshift*, since visible light is shifted toward the redder side of the spectrum (longer wavelengths). If $v < 0$ (source and observer moving toward each other), $\lambda' < \lambda$, meaning a *blueshift* (maybe it should be called a “violetshift,” but whatever).

Problem 2 (Energy Density of an Ideal Classical Non Relativistic Gas)

The number density n and energy density ρ of a gas of noninteracting particles (in $\hbar = c = k_B = 1$ units) are given by

$$n \equiv \frac{g}{(2\pi)^3} \int d^3\mathbf{p} f(\mathbf{p}), \quad \rho \equiv \frac{g}{(2\pi)^3} \int d^3\mathbf{p} E(\mathbf{p}) f(\mathbf{p}) \quad (11)$$

Here, g is the number of internal degrees of freedom of the particle, $E(\mathbf{p})$ is the energy of a particle with momentum $\mathbf{p}$, and $f(\mathbf{p})$ is the occupation number of a state with momentum $\mathbf{p}$. In the classical limit (a very dilute gas), we can use the Maxwell-Boltzmann distribution $f_{\text{MB}}(\mathbf{p}) = \exp(-[E(\mathbf{p}) - \mu]/T)$, where T is the temperature and μ is the chemical potential.

(a) Using the energy-momentum relationship of special relativity,

$$E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2} \quad (12)$$

write a Taylor expansion of $E(\mathbf{p})$ for nonrelativistic particles of mass m to order $\mathcal{O}(|\mathbf{p}|^2)$.

In natural units, the energy-momentum relationship in special relativity is $E^2 = |\mathbf{p}|^2 + m^2$, or

$$E(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2} \quad (13)$$

$$= m \sqrt{1 + \frac{|\mathbf{p}|^2}{m^2}} \quad \left(\text{nonrelativistic: } \frac{|\mathbf{p}|^2}{m^2} \ll 1 \right) \quad (14)$$

$$= m \left(1 + \frac{1}{2} \frac{|\mathbf{p}|^2}{m^2} + \dots \right) \quad (15)$$

$$= m + \frac{|\mathbf{p}|^2}{2m} + \dots \quad (16)$$

The first term is the rest energy; the second (and following) terms are the kinetic energy.

(b) Using your answer from part (a) and the definitions above, find the number density and energy density of a nonrelativistic classical gas in terms of m , T , μ , and g . You might find the following Gaussian integrals helpful:

$$\int_{-\infty}^{\infty} dx x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2}, \quad \int_{-\infty}^{\infty} dx x^4 e^{-x^2} = \frac{3\sqrt{\pi}}{4} \quad (17)$$

Plugging in our answer from part (a), we get

$$\begin{aligned} n &= \frac{g}{(2\pi)^3} \int d^3\mathbf{p} e^{-(E(\mathbf{p}) - \mu)/T} \\ &= \frac{g}{(2\pi)^3} \int d^3\mathbf{p} e^{-\left(m + \frac{|\mathbf{p}|^2}{2m} - \mu\right)/T} \\ &= \frac{g}{(2\pi)^3} e^{-(m - \mu)/T} \int d^3\mathbf{p} e^{-|\mathbf{p}|^2/(2mT)} \end{aligned} \quad (18)$$

Since the integrand only depends on the magnitude of $\mathbf{p}$, we can write $d^3\mathbf{p} = 4\pi|\mathbf{p}|^2 d|\mathbf{p}|$, which simplifies to

$$n = \frac{g}{2\pi^2} e^{-(m - \mu)/T} \int_0^{\infty} dp p^2 e^{-p^2/(2mT)} \quad (19)$$

Making the substitution $x \equiv p/\sqrt{2mT}$, we get

$$n = \frac{g}{2\pi^2} e^{-(m - \mu)/T} (2mT)^{3/2} \int_0^{\infty} dx x^2 e^{-x^2} \quad (20)$$

Since the integrand is even, we can write $\int_0^{\infty} dx x^2 e^{-x^2} = \frac{1}{2} \int_{-\infty}^{\infty} dx x^2 e^{-x^2}$ and then take the Gaussian integral:

$$n = \frac{g}{4\pi^2} e^{-(m - \mu)/T} (2mT)^{3/2} \int_{-\infty}^{\infty} dx x^2 e^{-x^2} \quad (21)$$

$$= \frac{g}{4\pi^2} e^{-(m - \mu)/T} (2mT)^{3/2} \left(\frac{\pi^{1/2}}{2} \right) \quad (22)$$

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m - \mu)/T} \quad (23)$$

We can do the same process with ρ :

$$\begin{aligned}
\rho &= \frac{g}{(2\pi)^3} \int d^3\mathbf{p} E(\mathbf{p}) e^{-(E(\mathbf{p})-\mu)/T} \\
&= \frac{g}{(2\pi)^3} \int d^3\mathbf{p} \left(m + \frac{|\mathbf{p}|^2}{2m} \right) e^{-\left(m + \frac{|\mathbf{p}|^2}{2m} - \mu\right)/T} \\
&= \frac{g}{(2\pi)^3} e^{-(m-\mu)/T} \left[m \int d^3\mathbf{p} e^{-|\mathbf{p}|^2/(2mT)} + \frac{1}{2m} \int d^3\mathbf{p} |\mathbf{p}|^2 e^{-|\mathbf{p}|^2/(2mT)} \right] \\
&= \frac{g}{2\pi^2} e^{-(m-\mu)/T} \left[m \int_0^\infty dp p^2 e^{-p^2/(2mT)} + \frac{1}{2m} \int_0^\infty dp p^4 e^{-p^2/(2mT)} \right] \\
&= \frac{g}{2\pi^2} e^{-(m-\mu)/T} \left[m(2mT)^{3/2} \int_0^\infty dx x^2 e^{-x^2} + \frac{(2mT)^{5/2}}{2m} \int_0^\infty dx x^4 e^{-x^2} \right] \\
&= \frac{g}{2\pi^2} e^{-(m-\mu)/T} \left[m(2mT)^{3/2} \left(\frac{\pi^{1/2}}{4} \right) + \frac{(2mT)^{5/2}}{2m} \left(\frac{3\pi^{1/2}}{8} \right) \right] \tag{24}
\end{aligned}$$

$$\rho = gm \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m-\mu)/T} + \frac{3}{2} gT \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m-\mu)/T} \tag{25}$$

(c) Write the energy density of a nonrelativistic classical gas in terms of m , T , and the number density of the gas n . In a previous physics or chemistry class, you might have seen the formula $E = \frac{3}{2}NkT$. Is your answer equivalent to that formula?

By directly pattern matching our answers from part (b), we can write

$$\rho = mn + \frac{3}{2}nT \tag{26}$$

If we multiply both sides by the volume, we get the total energy of the gas

$$U = \rho V = mN + \frac{3}{2}NT \tag{27}$$

The second term of this is the source of the formula $E = \frac{3}{2}NkT$, which you may have seen before. (We are using natural units, so temperatures are measured in energy units; consequently, Boltzmann's constant k_B does not appear in our version of the formula.) The first term is the rest energy of the gas. It can be ignored in the formula $E = \frac{3}{2}NkT$ since the rest energy does not change with temperature.

The average energy of a particle is given by

$$\begin{aligned}
E &= \frac{U}{N} \\
&= m + \frac{3}{2}T. \tag{28}
\end{aligned}$$